Exponential Growth and Decay

Math 130 - Essentials of Calculus

28 October 2019

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Similarly, it is possible to look at something whose value shrinks at a constant percentage: for example, say the value of a \$20000 car decreased by 15% per year, then the value after the first year would be

$$\$20000(1-.15) = \$20000(0.85)$$

and after t years it would be worth

$$\$20000(0.85)^t$$

Example

Example

A rare sculpture was purchased for $\$11.8$ *million and its value is expected to increase* 14% *per year.*

1. *Write an equation for the function that gives the value of the sculpture after* t *years.*
2. *What is the value of the sculpture after* 3.25 *years?*
3. *After how many years will the sculpture be worth* $\$20$ *million?*

Now You Try It!

Example

The eagle population in a state park is currently 1650 *but is expected to decrease* 18% *per year.*

1. *Write an equation for the function that gives the number of eagles in the park t years from now.*
2. *Determine the time required for the population to be reduced to* 1000.
3. *What is the rate of change of the population after four years?*

Compound Interest

If a savings account earns an annual interest rate of r (expressed as a decimal, not a percentage), then the future value of the account after t years with an initial investment of P dollars would be

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More typically, you will have a compounding period of less than a year, such as monthly or quarterly. If the compounding happens n times per year (e.g., $n = 4$ for quarterly compounding), then the interest rate per quarter will be given by $\frac{r}{n}$, where r is still given as a yearly rate. In this case, the future value after t years will be

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Compounding Continuously

It is also possible to increase the compounding to happen at every instant of time, which would correspond to taking the limit as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} A(t) &= \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} \\
&= \lim_{n\to\infty} P\left[\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} \\
&= P\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} \\
&= P\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{rt} \\
&= Pe^{rt}
\end{aligned}$$

Example

Example

If \$3000 is invested at 5% interest, find the value of the investment if interest is compounded

1. *annually*
2. *quarterly*
3. *monthly*
4. *continuously*

How long will it take for the value of the investment to double if the interest is compounded quarterly?

Exponential Growth

In the situation where a quantity changes at a constant percentage rate, we can model the situation with a *differential equation*:

$$A'(t) = k \cdot A(t)$$

which says that the rate of change in A is equal to k times A.

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$$A(t) = Ce^{kt}$$

where C is the *initial value* or initial quantity. k represents the *relative growth rate*.

Exponential Growth

Example

A bacteria culture initially contains 100 *cells and grows at a rate proportional to its size. After an hour, the population has increased to* 420.

1. *Find the relative growth rate.*
2. *Find an expression for the number of bacteria after t hours.*
3. *Find the number of bacteria after three hours.*
4. *Find the rate of growth after three hours.*
5. *When will the population reach* $10,000$?

Now You Try It!

Example

The half-life of the radioactive material cesium-137 is 30 *years. Suppose we have a* 100*mg sample.*

1. *Find the relative decay rate.*
2. *Write a formula that gives the mass that remains after t years.*
3. *How much of the sample remains after* 100 *years?*
4. *After how long will only* 1*mg remain?*
5. *At what rate is the mass decreasing after* 100 *years?*